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LOCALIZATION OF EQUIVARIANT COHOMOLOGY RINGS

BY

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ABSTRACT. The main result of this paper is the "calculation" of the Borel equivariant cohomology ring $H^*(EG \times_G X, \mathbf{Z}/p\mathbf{Z})$ localized at one of its minimal prime ideals. In case X is a point, the work of Quillen shows that the minimal primes $\mathfrak{p}_A$ are parameterized by the maximal elementary abelian p-subgroups A of G and the result is

$$H^*(BG, \mathbf{Z}/p\mathbf{Z})_{\mathfrak{p}_A} \cong H^*(BC_G(A), \mathbf{Z}/p\mathbf{Z})_{\mathfrak{p}_A}^{W_G(A)}.$$

Here, $C_G(A)$ is the centralizer of A in G, and $W_G(A) = N_G(A)/C_G(A)$, where $N_G(A)$ is the normalizer of A in G. An example is included.

1. Introduction and preliminaries. Let G be a compact Lie group and $EG \to BG$ a classifying bundle for principal G bundles. When G acts on a space X we may form the space $X_G = (EG \times X)/G = EG \times^G X$—the orbit space of $EG \times X$ under the diagonal action of G. Let p be a fixed prime; then we may consider the equivariant cohomology ring $H_G^*(X) = H^*(X_G, \mathbf{Z}/p\mathbf{Z})$ as a module over its commutative subring

$$H_G(X) = H(X_G, \mathbf{Z}/p\mathbf{Z}) = \begin{cases} \bigoplus_{i \geq 0} H^{2i}(X_G, \mathbf{Z}/p\mathbf{Z}), & p > 2, \\ \bigoplus_{i \geq 0} H^{i}(X_G, \mathbf{Z}/p\mathbf{Z}), & p = 2. \end{cases}$$

(Many of the general properties of equivariant cohomology are set forth in **[Q1, Q2, B, Br]** and various other sources and we shall not list these properties here. However, in order to be consistent with the above references and to suit the purposes of this paper, some restrictions must be placed on the types of G-spaces that we consider: they must be Hausdorff and either compact *or* paracompact with finite mod-p cohomological dimension (see, eg., **[Q1]** for a definition). In addition, we assume that every G-space has only a finite number of orbit types and that every orbit has a "slice" around it (see, e.g., **[Br]** for a discussion of these ideas).)

The main theorem of this paper (Theorem 3.2) "calculates" the localization of $H^*(X_G, \mathbf{Z}/p\mathbf{Z})$ at every *minimal* prime of $H(X_G, \mathbf{Z}/p\mathbf{Z})$. (These minimal primes were identified by Quillen in **[Q1, Q2]**.)

As an application of this theorem we calculate the localization of $H^*(\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z}), \mathbf{Z}/p\mathbf{Z})$ at one of the minimal primes of $H(\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z}), \mathbf{Z}/p\mathbf{Z})$.

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2. A localization lemma. Suppose X and Y are two spaces on which the compact Lie group G acts. When one gives $X \times Y$ the diagonal G-action, the projection π_X: $X \times Y \to X$ is G-equivariant and induces π_X^*: $H_G^*(X) \to H_G^*(X \times Y)$. If $G_{(x,y)}$ is the isotropy group at $(x, y) \in X \times Y$, then $G_{(x,y)}$ acts on the point x of X, inducing a map $H_G^*(X) \underset{\text{res}}{\to} H_{G_{(x,y)}}^*(\{x\})$. One also has a map

$$H_G^*(X) \underset{\alpha}{\to} H_G^*(G \cdot (x, y)) = H_G^*(Gx \times Gy)$$

(here $G \cdot (x, y)$, Gx, Gy, etc. denote orbits of the G-action) given by

$$H_G^*(X) \overset{\pi_X^*}{\to} H_G^*(X \times Y) \underset{\text{res}}{\to} H_G^*(G \cdot x \times G \cdot y).$$

Using the isomorphism $H_G^*(G \cdot (x, y)) \cong_\beta H_{G_{(x,y)}}^*(\{x\})$ induced by the chain of homeomorphisms

$$\begin{array}{ccccccc} EG \times G_{(x,y)}x & \leftrightarrow & EG/G_{(x,y)} & \leftrightarrow & (EG \times^G G)/G_{(x,y)} & \leftrightarrow & EG \times^G (G/G_{(x,y)}) \\ \| & & & & & & \swarrow\!\!\!\nearrow \\ EG_{(x,y)} \times^{G_{(x,y)}} \{x\} & & & & EG \times^G (Gx \times Gy) & & \end{array}$$

one sees immediately that there is a commutative triangle:

$$\begin{array}{ccc} H_G^*(X) & \overset{\alpha}{\to} & H_G^*(G \cdot x \times G \cdot y) \\ & \text{res} \searrow \quad \swarrow\!\!\!\nearrow \beta \cong & \\ & H_{G_{(x,y)}}^*(\{x\}) & \end{array}$$

Let $S \subseteq H_G(X)$ be a multiplicatively closed subset. Let

$$(X \times Y)^S = \left\{(x, y) | S \cap \ker\left(H_G^*(X) \overset{\text{res}}{\to} H_{G_{(x,y)}}^*(\{x\})\right) = \varnothing\right\}.$$

Using the above triangle, we see also that

$$(X \times Y)^S = \left\{(x, y) | S \cap \ker\left(H_G^*(x) \overset{\alpha}{\to} H_G^*(G \cdot (x, y))\right) = \varnothing\right\}.$$

Thus $(X \times Y)^S$ is G-invariant. It is also *closed*; we see this as follows.

Let

$$\begin{aligned} \mathscr{S}(G, X) &= \left\{H \leqslant G | X^H \neq \varnothing \text{ and } \exists x \in X^H \text{ with } S^{-1}H_H^*(\{x\}) = 0\right\} \\ &= \left\{H \leqslant G | X^H \neq \varnothing \text{ and } \forall x \in X^H, S^{-1}H_H^*(\{x\}) = 0\right\}. \end{aligned}$$

The set $\mathscr{S} = \mathscr{S}(G, X)$ is closed with respect to taking subconjugates; i.e., if $H \in \mathscr{S}$ and $K \lesssim H$, then $K \in \mathscr{S}$. For, if $K \lesssim H$, then X^H is contained in a translate of X^K; so if $X^H \neq \varnothing$, then $X^K \neq \varnothing$. Also, the diagram

$$\begin{array}{ccc} & H_H^*(\{x\}) \xrightarrow{\text{res}_2} & H_{gKg^{-1}}^*(\{x\}) \\ \text{res}_1 \nearrow & & \uparrow c_g \cong \\ H_G^*(X) & \xrightarrow{\text{res}_3} \ (\text{to } H_{gKg^{-1}}^*(\{x\})) & H_K^*(\{g^{-1}x\}) \\ \uparrow c_g = \text{id} & & \\ H_G^*(X) & \xrightarrow{\text{res}_4} \ (\text{to } H_K^*(\{g^{-1}x\})) & \end{array}$$

commutes, if $x \in X^H$ and $g \in G$; thus

$$S \cap \ker(\mathrm{res}_1) \subseteq S \cap \ker(\mathrm{res}_2 \circ \mathrm{res}_1) = S \cap \ker(\mathrm{res}_3)$$
$$= S \cap \ker\left(c_g^{-1} \circ \mathrm{res}_3\right) = S \cap \ker(\mathrm{res}_4).$$

Therefore, $S^{-1}(H_H^*(\{x\})) = 0$ and $S^{-1}H_K^*(\{g^{-1}x\}) = 0$ if $gKg^{-1} \not\leq H$. We note that $(X \times Y)^S$ can be characterized as the *smallest* subset of $X \times Y$ with the property that every isotropy group of its complement lies in $\mathscr{S}$. Thus if $\xi \in (X \times Y) - (X \times Y)^S$, then $G_\xi \in \mathscr{S}$. By the slice theorem, there is an open invariant neighborhood U of ξ such that $G_u \subseteq G_\xi$ for every $u \in U$; so $G_u \in \mathscr{S}$ for every $u \in U$. So all isotropy of U is in $\mathscr{S}$; i.e. all isotropy of the complement of $(X \times Y) - U$ is in $\mathscr{S}$. Thus $(X \times Y)^S \subseteq (X \times Y) - U$ so $U \subseteq (X \times Y) - (X \times Y)^S$, proving that $(X \times Y)^S$ is closed.

One has the following localization result (compare with tom Dieck [T] and Hsiang [H]).

LEMMA 2.1. *Suppose either*:

(i) *$X \times Y$ is compact and every orbit in X is a G-deformation retract of one of its open neighborhoods, or*

(ii) *$X \times Y$ is paracompact with finite* mod-*p cohomology dimension.*

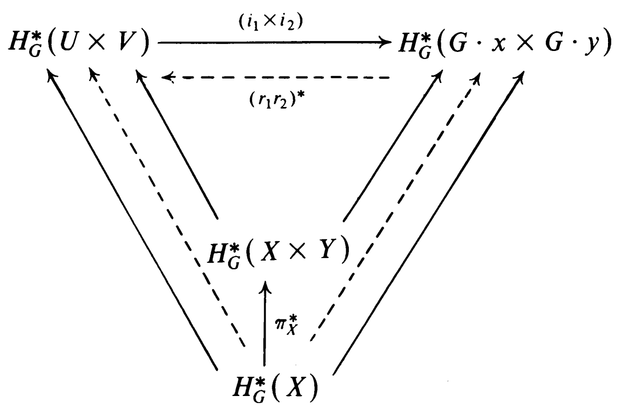
If Z is any closed invariant subset of $X \times Y$ such that all of the isotropy of $(X \times Y) - Z$ lies in $\mathscr{S}(G, X)$, then there is an isomorphism

$$S^{-1}H_G^*(X \times Y) \underset{\cong}{\overset{S^{-1}\mathrm{res}}{\to}} S^{-1}H_G^*(Z).$$

PROOF. This is a straightforward modification of the standard localization theorems of, for example, tom Dieck [T] and Hsiang [H]. (Note that their results are obtained by letting $Z = (X \times Y)^S$ and $X = \mathfrak{pt}$.)

We give the proofs of both (i) and (ii) for completeness.

(i) For every point (x, y) in $X \times Y$, there is by assumption an invariant neighborhood U of $G \cdot x$ and a *G-deformation* retraction r_1: $U \to G \cdot x$. By existence of slice, of course, there is an invariant neighborhood V of $G \cdot y$ and an equivariant retraction r_2: $V \to G \cdot y$. Consider the following diagram (in which the diagram of solid arrows commutes):



The key point in the argument is to note that the diagram of *dotted* arrows commutes:

$$\begin{aligned}(r_1 \times r_2)^* i^* \pi_X^* &= (r_1 \times r_2)^*(i_2 \times i_2)^* j^* \pi_X^* = (i_1 r_1 \times i_2 r_2)^* j^* \pi_X^* \\ &= (\mathrm{id} \times i_2 r_2)^* j^* \pi_x^* \quad (\text{since } i_1 r_1 \cong_G \mathrm{id}) \\ &= [\pi_X j(\mathrm{id} \times i_2 r_2)]^* = (\pi_X j)^* = j^* \pi_X^*.\end{aligned}$$

Thus $\ker \alpha = \ker \alpha'$ since $(r_1 \times r_2)^*$ is injective. Now we use the standard arguments to prove (i).

Suppose W is a closed invariant subset of $X \times Y$ with the property that all of its isotropy is in $\mathscr{S}$. For each $(x, y) \in W$, therefore, there is an $s(x, y) \in S$ such that $s(x, y) \in \ker \alpha$. Choosing retractive invariant neighborhoods (in $X \times Y$) U_x, U_y (of $G \cdot x, G \cdot y$, respectively) as above we see that $S^{-1}H_G^*(U_x \times U_y) = 0$ since $s(x, y) \in \ker \alpha' = \ker \alpha$. Cover W by a finite number of such neighborhoods and then use a straightforward Mayer-Vietoris argument, plus exactness of localization, to find an invariant neighborhood $\mathscr{O}$ of W with $S^{-1}H_G^*(\mathscr{O}) = 0$. Now use the continuity property (e.g. see **[Q1]**) and the fact that localization commutes with direct limits to show that $S^{-1}H_G^*(W) = 0$.

Considering Z as in the theorem, we fix a closed invariant neighborhood V of Z and use the above argument applied to the Mayer-Vietoris sequence for $(X \times Y) = V \cup ((X \times Y) - \mathrm{int}\, V)$ plus exactness of localization, to get

$$S^{-1}H_G^*(V) \cong S^{-1}H_G^*(X \times Y).$$

Since Z is closed we may let V vary and again use the continuity property and the fact that localization commutes with direct limits to conclude that $S^{-1}H_G^*(X \times Y) \cong S^{-1}H_G^*(Z)$.

(ii) Again, as in the proof of (i), suppose that W is a closed invariant subset of $X \times Y$ with the property that all of its isotropy is in $\mathscr{S}$. W is paracompact since it is a closed subset of $X \times Y$. Also, W has finite mod-p cohomological dimension. Thus the mod-p cohomological dimension of W/G is finite **[Q1]** and the E_2 term of the Leray spectral sequence

$$H^p(W/G, \mathscr{H}^q) \Rightarrow H_G^{p+q}(W)$$

is bounded from the right; i.e., $E_2^{p,q} = 0$ for $p > N$. Here, the stalk at $G \cdot w \in W/G$ of $\mathscr{H}^q$ is $H_{G_s}^q = H_G^1(G \cdot w)$. Now, since all of the isotropy of W is in $\mathscr{S}$, for each $w \in W$, one has an $s(w) \in S$ such that $s(w) \in \ker(H_G^*(X) \xrightarrow{\alpha} H_G^*(Gw))$. Since $X \times Y$ and (hence) W have only a finite number of orbit types $\{G \cdot w_1, \dots, G \cdot w_n\}$, it is not hard to see that

$$s = s(w_1) \cdots s(w_n) \in \left[\bigcap_{w \in W} \ker(H_G^*(X) \to H_G^*(G \cdot w))\right] \cap S.$$

The $H_G^*(X)$-module structure on the E_2 term (given by π_X^* and restriction on the coefficients $\mathscr{H}^q$) passes to the limit and is the same at the limit (at least up to sign) as the $H_G^*(X)$-module structure induced there by the $H_G^*(X)$-module structure $H_G^*(W)$ (again given by π_X^* and restriction).

Therefore s maps to 0 in every stalk of the sheaf H_G^* and hence maps to 0 in $H^0(W/G, \mathscr{H}^*) = E_2^{0,*}$ and in $E_\infty^{0,*}$.

Following Hsiang's argument [**H**] through, one may conclude that s^{N+1} maps to zero in $H_G^*(W)$ and therefore that $S^{-1}H_G^*(W) = 0$. We now finish the argument exactly as in case (i). Q.E.D.

From now on, we assume that X is either compact and every orbit of X is a G-deformation retract of one of its neighborhoods, *or* that X is paracompact with finite mod-p cohomological dimension.

3. The localization of $H_G^*(X, \mathbf{Z}/p\mathbf{Z})$ at the minimal primes of $H_G(X, \mathbf{Z}/p\mathbf{Z})$. The minimal primes of $H_G(X, \mathbf{Z}/p\mathbf{Z})$ were described by Quillen [**Q1, Q2**] as follows. An *elementary abelian p-subgroup* or *p-torus* in G is a subgroup A of G which is a direct product of cyclic groups of order p. The number of cyclic factors is the *rank* of A.

Let $\mathscr{A}(G, X)$ denote Quillen's category of pairs [**Q2**] (A, c) where A is a p-torus in G and C is a (nonempty) connected component of X^A (the fixpoint set of the A-action on X); i.e., the objects of $\mathscr{A}(G, X)$ are the pairs (A, c) as above, and a morphism θ: $(A, c) \to (A', c')$ is a conjugation of A into (a subgroup of) A' by an element g of G such that $c' \subseteq gc$ as well. The objects of $\mathscr{A}(G, X)$ are partially ordered; i.e., $(A, c) \leqslant (A', c')$ if and only $A \leqslant A'$ and c is the unique component of X^A containing C'. We say that (A, c) is *subconjugate* to (A', c') (written $(A, c) \lesssim (A', c')$) if $\mathrm{Hom}_{\mathscr{A}(G,X)}((A, c), (A', c')) \neq \varnothing$; and (A, c) is *conjugate* to $(A', c')((A, c) \cong (A', c'))$ if (A, c) and (A', c') are isomorphic objects.

For every pair (A, c) in $\mathscr{A}(G, X)$, there is a *prime* ideal $\mathfrak{p}_{(A,c)} \subseteq H(X_G, \mathbf{Z}/p\mathbf{Z})$; namely

$$\mathfrak{p}_{(A,c)} = \ker\left(H_G(X) \overset{\text{res}}{\to} H_A(\mathfrak{p}\mathrm{t})/\sqrt{0} \right);$$

where $\mathfrak{p}\mathrm{t} \in X^A$ is any point in X^A. One has

THEOREM 3.1 (PROPOSITION 11.2 OF [**Q2**]). (i) *$\mathfrak{p}_{(A,c)} \supseteq \mathfrak{p}_{(A',c')}$ if and only if $(A, c) \lesssim (A', c')$; in particular, $\mathfrak{p}_{(A,c)} = \mathfrak{p}_{(A',c')}$ if and only if $(A, c) \cong (A', c')$.*

(ii) *There is a one-to-one correspondence between conjugacy classes of maximal pairs (A, c) and minimal prime ideals of $H_G(X)$:*

$$[(A, c)] \mapsto \mathfrak{p}_{(A,c)}.$$

One may define (as in [**Q2**]) $C_G(A, c) = \{g \in G | ga = ag \ \forall a \in A \text{ and } c = c\}$; $N_G(A, c) = \{g \in G | gAg^{-1} = A \text{ and } gc = c\}$, and $W_G(A, c) = W_G(A, c) = N_G(A, c)/C_G(A, c)$. There is a natural action of $W_G(A, c)$ on $H^*_{C_G(A,c)}(c)$; we may consider both the invariant subring $H^*_{C_G(A,c)}(c)^{W_G(A,c)}$, of this action, *and* the ring $H^*_{N_G(A,c)}(c)$ as $H_G^*(X)$-modules via restriction. Hence, we may localize all of these modules with respect to the prime $\mathfrak{p}_{(A,c)}$ of $H_G(X)$. The main theorem of this paper is

THEOREM 3.2. *Suppose that (A, c) is a maximal pair of $\mathscr{A}(G, X)$. Then there is an isomorphism*

$$H_G^*(X)_{\mathfrak{p}_{(A,c)}} \xrightarrow{(\text{res})_{\mathfrak{p}_{(A,c)}}} H^*_{C_G(A,c)}(c)^{W_G(A,c)}_{\mathfrak{p}_{(A,c)}}.$$

We note that $\operatorname{res}(H_G^*(X)) \subseteq H^*_{C_G(A,c)}(c)^{W_G(A,c)}$ since inner automorphisms act trivially on equivariant cohomology (e.g., see **[Q1]**), so $C_G(A,c)$ acts trivially on $H_G^*(X)$.

We must first prove some preliminary lemmas. Fix an embedding of G in a unitary group U; let S be the "diagonal" p-torus of U and let $F = U/S$ be the compact smooth G-manifold of right cosets of S in U.

LEMMA 3.3. *Suppose that (A, c) is a maximal pair in $\mathscr{A}(G, X)$. Let K be a closed subgroup of G, and let W be a closed K-invariant subset of X such that (A, c) is also an object in the category $\mathscr{A}(K, W)$. Then there are localization isomorphisms for every $i \geqslant 1$,*

$$(*) \qquad S^{-1}H_K^*(W \times F^i) \underset{\cong}{\rightarrow} S^{-1}H_K^*\big(G \cdot ((c \cap w) \times (F^A)^i)\big),$$

where $S = \theta(H_G(X) - \mathfrak{p}_{(A,c)})$ and θ is the restriction $H_G(X) \to H_K(W)$.

Moreover, if one also knows that any pair in $\mathscr{A}(K, W)$ equivalent to (A, c) in $\mathscr{A}(G, X)$ is also equivalent to (A, c) in $\mathscr{A}(K, W)$ (we then say (A, c) is weakly closed in $\mathscr{A}(K, W)$ with respect to $\mathscr{A}(G, X)$), then there are isomorphisms for every $i \geqslant 1$,

$$(**) \qquad S^{-1}H_K^*(W \times F^i) \overset{\cong}{\rightarrow} S^{-1}H_K^*\big(K \cdot (c \times (F^A)^i)\big).$$

PROOF. We apply the localization theorem to W (not X) with $Y = F^i$. (W inherits the necessary properties of X since it is closed. $W \times F^i$ then has the necessary properties since F^i is a compact manifold.)

We need to verify

(i) that $G \cdot (c \cap W \times (F^A)^i)$ and $K \cdot (c \times (F^A)^i)$ are closed K-invariant subsets of $W \times F^i$ and

(ii) that ll of the isotropy of the complement of $G \cdot (c \cap W \times (F^A)^i)$ (or of $K \cdot (c \times (F^A)^i)$ in case (A, c) is weakly closed) is in

$$\mathscr{S}(K, W) = \{K' \leqslant K \mid W^{K'} \neq \varnothing \text{ and } S^{-1}H_{K'}^*(\{w\}) = 0\ \forall w \in W^{K'}\}.$$

(i) is straightforward.

To show (ii), suppose that $\xi \in W \times F^i$ and $K_\xi \notin \mathscr{S}(K, W)$. Let $\xi = (x, f_1, \ldots, f_i)$. Then $K_\xi = K_x \cap K_{f_1} \cap \cdots \cap K_{f_i} = B$ is a p-torus in K since all isotropy groups of F are p-tori **[Q1]**. Also, $x \in W^B$ since $K_x \geqslant K_\xi = B$. Let us suppose that x lies in the component d of W^B, which in turn lies in the component c' of X^B.

Since $K_\xi \notin \mathscr{S}(K, W)$, $S^{-1}H_B^*(\{x\}) \neq 0$; i.e. $S \cap \ker(H_k^*(W) \to H_B^*(\{x\})) = \varnothing$. But this means that

$$S \cap \ker\big(H_K(W) \to H_B(x \in d)/\sqrt{0} = H_B(x \in c')/\sqrt{0}\big) = \varnothing.$$

Consider the following commutative diagram of restrictions:

$$\begin{array}{ccc} H_G(X) & \searrow & \\ \Big\downarrow \theta & & H_B(x \in d)/\sqrt{0} = H_B(x \in c')/\sqrt{0}. \\ H_K(W) & \nearrow & \end{array}$$

Since $\theta(H_G(X) - \mathfrak{p}_{(A,c)}) \cap \ker[H_K(W) \to H_B(x \in d)/\sqrt{0}\,] = \varnothing$ the above diagram shows that $[H_G(X) - \mathfrak{p}_{(A,c)}] \cap \ker(H_G(X) \to H_B(x \in x)/\sqrt{0}\,) = \varnothing$. Thus $\mathfrak{p}_{(B,c')} \subseteq \mathfrak{p}_{(A,c)}$ in $H_G(X)$. Therefore, since $\mathfrak{p}_{(A,c)}$ is minimal **[Q2]**, $\mathfrak{p}_{(B,c')} = \mathfrak{p}_{(A,c)}$. But then $(B, c') \cong (A, c)$ in $\mathscr{A}(G, X)$ **[Q2]**. By definition, this means that there is some $g \in G$ such that $gBg^{-1} = A$ and $gc' = c$. Since $x \in c'$, $x \in gc' = c$; and since $f_j \in F^B$ for each j (since $B \subseteq K_{f_j}$ for every j), we see that $fg_j \in gF^B = F^{gBg^{-1}} = F^A$ so that $g\xi \in c \times (F^A)^i$ and thus that $\xi \in (Gc \cap W) \times G \cdot (F^A)^i$.

For (**), we simply note that if the g in the paragraph above may always be chosen from K, then $K_\xi \notin \mathscr{S}(K, W)$ implies that $\xi \in K \cdot (c \times (F^A)^i)$. Q.E.D.

LEMMA 3.4. *If (A, c) is a maximal pair in (G, X), then there are isomorphisms for $i \geqslant 1$,*

$$H_G^*\big(G \cdot \big(c \times (F^A)^i\big)\big) \overset{\cong}{\to} H_{N_G(A,c)}^*\big(c \times (F^A)^i\big).$$

PROOF. This follows from the fact there is a G-equivariant homeomorphism

$$G \times^{N_G(A,c)} \big[c \times (F^A)^i\big] \overset{\theta}{\to} G \cdot \big(c \times (F^A)^i\big),$$

where $\theta([g,(x, f_1,\ldots,f_i)]) = (gx, gf_1,\ldots,gf_i)$. Here, $N_G(A, c)$ acts on G by $g \overset{n}{\mapsto} gn^{-1}$; and on $c \times (F^A)^i$ diagonally on the left: $(x, f_1,\ldots,f_i) \overset{n}{\mapsto} (nx, nf_1,\ldots,nf_i)$. The map θ is clearly surjective, continuous and open. If $(gx, gf_1,\ldots) = (g'x', g'f_1',\ldots)$ where x and x' are in c, and $f_1,\ldots; f_1',\ldots;$ are in F^A, then $gx = g'x'$ and $gf_j = g'f_j'$. Thus if $h = g^{-1}g'$, $hx' = x$ and $hf_1' = f_1,\ldots,$ etc., so that $G_x = hG_{x'}h^{-1}$ and $G_{f_1} = hG_{f_1'}h^{-1}$. Now, since $x \in c \subseteq X^A$ and $f_1 \in F^A$, $G_x \supseteq A$ and $G_{f_1} \supseteq A$. So $A \subseteq G_x \cap G_{f_1}$. Since all isotropy of F is p-toral, G_{f_1} is a p-torus of G; thus $B = G_x \cap G_{f_1}$ is a p-torus in G. Since B fixes x and $A \subseteq B$, $X^B \subseteq X^A$ and thus there is a component d of X^B containing x and contained in c. Thus $(A, c) \subseteq (B, d)$ in $\mathscr{A}(G, X)$, so by maximality, $(A, c) = (B, d)$.

Similarly, if $B' = G_{X'} \cap G_{f_1'}$ we see that $(A, c) = (B', d')$ where d' is a component of $X^{B'}$ containing x and contained in c. Thus $B = B' = A$ and $d = d' = c$. But $hB'h^{-1}B$ and $hd' = d$, so $hAh^{-1} = A$ and $hc = c$. Therefore, $h \in N_G(A, c)$ and clearly

$$\big[\big(gh, h^{-1}x,\ldots,h^{-1}f_i\big)\big] = \big[\big(g', x', f_1',\ldots,f_i'\big)\big]$$

in $G \times^{N_G(A,c)}(c \times (F^A)^i)$. So θ is a homeomorphism yielding a homeomorphism:

$$\begin{array}{l} EG \times^G \big(G \times^{N_G(A,c)} \big[c \times (F^A)^i\big]\big) \overset{\theta_G}{\to} EG \times^G \big(G \cdot \big(c \times (F^A)^i\big)\big) \\ \qquad\qquad \| \\ [EG \times^G G] \times^{N_G(A,c)} \big[c \times (F^A)^i\big] \\ \qquad\qquad \| \\ EG \times^{N_G(A,c)} \big[c \times (F^A)^i\big] = E(N_G(A,c)) \times^{N_G(A,c)} \big[c \times (F^A)^i\big]. \end{array}$$

Q.E.D.

Before we proceed we must look more closely at the manifold F and its submanifolds F^B where B is a p-torus of G. Recall that $F = U(n)/S(n)$ where $U(n)$ is a unitary group (for an n-dimensional complex vector space) in which G is embedded. The diagonal p-torus of $U(n)$ is $S(n)$. Thus a point in $F = U/S$ may be described as a pair $\{(l_1,\dots,l_n);\ (v_1^*,\dots,v_n^*)\}$ where (l_i) is an ordered n-tuple of mutually orthogonal lines and v_i^* is the orbit of a vector v_i in the unit sphere $S^1(l_i)$ in the line l_i under the action of the group of pth roots of unity in this sphere. The action of G on F is via the embedding in U; i.e., one thinks of $g \in G$ as a unitary transformation of C^n, thus

$$g \cdot \{(l_n,\dots,l_n);(v_1^*,\dots,v_n^*)\} = \{(gl_1,\dots,gl_n);((gv_1)^*,\dots,(gv_n)^*)\}.$$

(This makes sense, since g preserves lengths and angles—the action of the group of pth roots of unity is given by rotation through a fixed angle.)

Now, suppose that A is a p-torus in G. The embedding $A \hookrightarrow G \hookrightarrow U$ gives us a unitary representation of A on $\mathbf{C}^n$ (the restriction of the representation $G \hookrightarrow U$ of G). This representation may be decomposed according to a set of distinct n-dimensional complex irreducible characters $\{\chi_1,\dots,\chi_k\}$ of A. If we let V_j denote the eigenspace of the character χ_j and $n_j = \dim V_j$, then we have an orthogonal decomposition of $\mathbf{C}^n$,

$$\mathbf{C}^n = V_1 \perp \cdots \perp V_k \quad \text{and} \quad n = n_1 + \cdots + n_k.$$

(We assume that $n_j > 0$ for every j.)

Let $\Sigma(n_1,\dots,n_k)$ be the set of $(n_1,\dots,n_k)$-shuffles of $\{1,2,\dots,n\}$;

$$\Sigma(n_1,\dots,n_k) = \big\{\sigma \in \Sigma(n) | \sigma(i_{11}, i_{12},\dots,i_{1n_1}, i_{21},\dots,i_{2n_2},\dots,i_{k1},\dots,i_{kn_k}) \\ \text{and } i_{j1} < i_{j2} < \cdots < i_{jn_j}\ \forall j \in \{1,2,\dots,k\}\big\}.$$

We claim that

$$(*) \qquad F^A = \coprod_{\sigma \in \Sigma(n_1,\dots,n_k)} \sigma \cdot (F(V_1) \times \cdots \times F(V_k)).$$

Here, $F(V_i)$ is the flag space for V_i, i.e., $F(V_i)$ consists of the pairs

$$\big\{\big(l_{n_1+\cdots+n_{i-1}+1},\dots,l_{n_1+\cdots+n_i}\big);\ \big(v^*_{n_1+\cdots+n_{i-1}+1}, v^*_{n_1+\cdots+n_i}\big)\big\}$$

where the l_j's are an ordered set of n_i orthogonal lines in V_i, and V_j^*'s are orbits of vectors V_j in the unit spheres of the l_j's under the group(s) of pth roots of unity in these spheres.

It should be clear what subset of F that $\sigma \cdot (F(V_1) \times \cdots \times F(V_k))$ denotes: if $\sigma = (i_{11},\dots,i_{1n_1}; i_{12},\dots,i_{2n_2};\dots; i_{k1},\dots,i_{kn_k})$, then

$$\sigma \cdot \Big\{\Big(\underbrace{l_1,\dots,l_{n_1}}_{\text{in } V_1};\ \underbrace{l_{n_1+1},\dots,l_{n_1+n_2};\dots}_{\text{in } V_2}\Big);\ (v_1^*,\dots,v_{n_1}^*;\dots)\Big\}$$

$$= \big\{\big(l_{i_{11}}, l_{i_{12}},\dots,l_{i_1};\ l_{i_{21}},\dots,l_{i_{2n_2}};\dots\big);\ \big(v^*_{i_{11}},\dots,v^*_{i_{1n_1}};\dots\big)\big\}.$$

Note that if $\sigma \neq \tau$ then $\sigma \cdot (F(V_1) \times \cdots \times F(V_k)) \cap \tau \cdot (F(V_1) \times \cdots \times F(V_k))$ is indeed empty.

Since *every* line represented in the direct sum on the right in equation (*) is an eigenspace for some character $\chi_{\alpha'}$, this line (and every *orbit* in its unit sphere) is fixed by everything in A. On the other hand, if an ordered collection of lines (and orbits) in F are fixed by everything in A, then each line must be an eigenspace for the A-action, thus contained in some V_i. So equality (*) holds.

Since $F(V_1)\times\cdots\times F(V_k)$ is clearly connected and closed and each $\sigma\cdot(F(V_1)\times\cdots\times F(V_k))$ is homeomorphic to $F(V_1)\times\cdots\times F(V_k)$, the connected components of F^A are the $\sigma\cdot(F(V_1)\times\cdots\times F(V_k))$.

Now, $C_G(A)$ fixes each component of F^A. It is enough to show that $C_G(A)$ fixes $F(V_1)\times\cdots\times F(V_k)$. For this, we need only show that if l is a line in V_j then gl is also a line in V_j if $g\in C_G(A)$. (The orbits are carried along without difficulty.) Let $l=\mathbf{C}\cdot\vec{x}$ where $\vec{x}$ is a vector in V_j. Then $g\vec{x}$ is an eigenvector for χ_j: $b\cdot g\vec{x}=gb\vec{x}=g\cdot\chi_j(b)\cdot\vec{x}=\chi_j(b)\cdot g\vec{x}$, if $b\in A$.

Conversely, we want to show that if $g\in N_G(A)$, and g fixes a component of F^A, then $g\in C_G(A)$. Again, it is enough to show this for the component $F(V_1)\times\cdots\times F(V_k)$, so let $g\in N_G(A)$ be such that

$$g\cdot(F(V_1)\times\cdots\times F(V_k))=F(V_1)\times\cdots\times F(V_k).$$

Then g carries lines in V_j to lines in V_j $\forall j$, so if $\vec{x}\in V_j$, $g\vec{x}\in V_j$ and then $b\cdot g\vec{x}=\chi_j(b)g\vec{x}=g\cdot\chi_j(b)\vec{x}=g\cdot b\vec{x}$ for every $b\in A$, so $b^{-1}g^{-1}bg$ fixes every vector in every V_j. But $G\hookrightarrow U$ so that $b^{-1}g^{-1}bg=\mathrm{id}=\Rightarrow bg=gb$ and $g\in C_G(A)$. (One needs $g\in N_G(A)$ to ensure that $g\cdot F^A\subseteq F^A$.) Thus one sees from the above picture of F^A that $W_G(A)=N_G(A)/C_G(A)$ acts *freely* on $\pi_0(F^A)$. Using this picture of F^A (pointed out to me by Quillen, who also pointed out the following lemma) one may conclude

LEMMA 3.5. *If (A,c) is a pair in $\mathscr{A}(G,X)$ then $H^q_{C_G(A,c)}(c\times(F^A)^i)$ is a free $Z/pZ[W_G(A,c)]$-module for every $q\geqslant 0$ and $i\geqslant 1$.*

PROOF. $W_G(A,c)$ acts freely on $\pi_0(c\times(F^A)^i)=\{c\}\times\pi_0(F^A)^i$, so it acts freely on

$$H^q_{C_G(A,c)}\big(c\times(F^A)^i\big)=\bigoplus_{\xi\in\pi_0(c\times(F^A)^i)}H^q_{C_G(A,c)}(\xi).\quad \text{Q.E.D.}$$

LEMMA 3.6. *If (A,c) is a pair in $\mathscr{A}(G,X)$ then there are isomorphisms for $i\geqslant 1$,*

$$H^*_{N_G(A,c)}\big(c\times(F^A)^i\big)\xrightarrow{\cong}H^*_{C_G(A,c)}\big(c\times(F^A)^i\big)^{W_G(A,c)}$$

PROOF. Let $N=N_G(A,c)$, $C+C_G(A,c)$, $W=N/C$ and $Z=c\times(F^A)^i$ for a fixed $i\geqslant 1$.

Then Z_C is a principal W-bundle over Z_n and there is a Serre spectral sequence **[B]**

$$H^p(BW,\{H^q_C(z)\})\Rightarrow H^{p+q}_N(Z).$$

Now, W is a finite group, so we may use results from the theory of cohomology of finite groups to compute $H^p(BW,\{H^q_C(Z)\})$. By Lemma 3.5, $H^q_C(Z)$ is a *free* $Z/pZ[W]$-module for $q\geqslant 0$, so one may conclude **[C-E]** that

$$H^p(BW,\{H^q_C(Z)\})=0\quad\text{if } p>0,\ q\geqslant 0.$$

Thus the above spectral sequence degenerates yielding

$$H^0\Big(BW, \big\{ H_C^*\big(c \times (F^A)^i\big)\big\}\Big) \overset{\cong}{\leftarrow} H_N^*\big(c \times (F^A)^i\big)$$

or

$$H_C^*\big(c \times (F^A)^i\big)^W \overset{\cong}{\leftarrow} H_N^*\big(c \times (F^A)^i\big). \quad \text{Q.E.D.}$$

Finally, we return to the

PROOF OF THEOREM 3.2. Consider the following commutative diagram (it should be clear what the maps are):

$$\begin{array}{llcl}
(*) \quad H_G^*(X)_{\mathfrak{p}_{(A,c)}} \to & H_G^*(X \times F)_{\mathfrak{p}_{(A,c)}} & \rightrightarrows & H_G^*(X \times F^q)_{\mathfrak{p}_{(A,c)}} \\
& \downarrow ① & & \downarrow \\
& H_G^*\big(G \cdot (c \times F^A)\big)_{\mathfrak{p}_{(A,c)}} & \rightrightarrows & H_G^*\Big(G \cdot \big(c \times (F^A)^2\big)\Big)_{\mathfrak{p}_{(A,c)}} \\
& \downarrow ② & & \downarrow \\
& H_{N_G(A,c)}^*(c \times F^A)_{\mathfrak{p}_{(A,c)}} & \rightrightarrows & H_{N_G(A,c)}^*\big(c \times (F^A)^2\big)_{\mathfrak{p}_{(A,c)}} \\
& \downarrow ③ & & \downarrow \\
& H_{C_G(A,c)}^*(c \times F^A)_{\mathfrak{p}_{(A,c)}}^{W_G(A,c)} & \rightrightarrows & H_{C_G(A,c)}^*\big(c \times (F^A)^2\big)_{\mathfrak{p}_{(A,c)}}^{W_G(A,c)} \\
& \uparrow ④ & & \uparrow \\
& H_{C_G(A,c)}^*(c \times F)_{\mathfrak{p}_{(A,c)}}^{W_G(A,c)} & \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} & H_{C_G(A,c)}^*(c \times F^2)_{\mathfrak{p}_{(A,c)}}^{W_G(A,c)}.
\end{array}$$

We observe:

(i) the sequence $(*)$ is exact: this follows from the exact sequence of **[Q1]** plus exactness of localization;

(ii) the vertical arrows in each of squares 1 through 4 are isomorphisms:

Square 1. Apply Lemma 3.3 with $W = X$ and $K = G$.

Square 2. Apply Lemma 3.4, then localize.

Square 3. Apply Lemma 3.6, then localize.

Square 4. Apply Lemma 3.3 with $W = c$ and $K = C_G(A, c)$.

Since (A, c) is maximal, it is "weakly closed" in $\mathscr{A}(K, W)$ in this case, and the isomorphism is isomorphism $(**)$ of Lemma 3.3.

Thus we conclude that $\ker(\beta_1, \beta_2) \cong H_G^*(X)_{\mathfrak{p}_{(A,c)}}$. But, the exact sequence

$$H_{C_G(A,c)}^*(c) \to H_{C_G(A,c)}^*(c \times F) \rightrightarrows H_{C_G(A,c)}^*(c \times F^2)$$

of **[Q1]** plus exactness of localization of left exactness of taking invariants enables us to conclude that

$$H_{C_G(A,c)}^*(c)_{\mathfrak{p}_{(A,c)}}^{W_G(A,c)} \underset{\cong}{\leftarrow} H_G^*(X)_{\mathfrak{p}_{(A,c)}}. \quad \text{Q.E.D.}$$

Let $H_G^*(X) \underset{r_{G,N}}{\to} H_{N_G(A,c)}^*(c)$ be the restriction map. If

$$\mathfrak{p}_{(A,c)} = \ker\big(H_G(X) \to H_A(\mathfrak{p}\mathrm{t} \in c)/\sqrt{0}\,\big)$$

as usual, and

$$\mathfrak{p}^N_{(A,c)} = \ker\Big(H_{N_G(A,c)}(c) \to H_A(\mathrm{pt} \in c)/\sqrt{0} \Big)$$

then $r^{-1}_{G,N}(\mathfrak{p}^N_{(A,c)}) = \mathfrak{p}_{(A,c)}$. Similarly, there is the prime $\mathfrak{p}^C_{(A,c)} \subseteq H_{C_G(A,c)}(c)$ and the restriction $H_G(X) \underset{r_{G,C}}{\to} H_{C_G(A,c)}(c)$ with $r^{-1}_{G,C}(\mathfrak{p}^C_{(A,c)}) = \mathfrak{p}_{(A,c)}$. Also we have $r^{-1}_{N,C}(\mathfrak{p}^C_{(A,c)}) = \mathfrak{p}^N_{(A,c)}$ where $H_{N_G(A,c)}(c) \underset{r_{N,C}}{\to} H_{C_G(A,c)}(c)$.

COROLLARY 3.7. *If (A, c) is a maximal pair of $\mathscr{A}(G, X)$, then*

$$H^*_G(X)_{\mathfrak{p}_{(A,c)}} \overset{\cong}{\to} H^*_{N_G(A,c)}(c)_{\mathfrak{p}^N_{(A,c)}}.$$

PROOF. The main theorem gives an isomorphism

$$H^*_G(X)_{\mathfrak{p}_{(A,c)}} \underset{\cong}{\to} H^*_{C_G(A,c)}(c)^{W_G(A,c)}_{\mathfrak{p}_{(A,c)}}.$$

Applied to $N_G(A, c)$ and c instead of G and X, it also gives an isomorphism

$$H^*_{N_G(A,c)}(c)_{\mathfrak{p}^N_{(A,c)}} \underset{\cong}{\to} H^*_{C_G(A,c)}(c)^{W_G(A,c)}_{\mathfrak{p}^N_{(A,c)}}.$$

Thus we need only show that

$$H^*_{C_G(A,c)}(c)^{W_G(A,c)}_{\mathfrak{p}^N_{(A,c)}} \cong H^*_{C_G(A,c)}(c)^{W_G(A,c)}_{(A,c)}$$

or

$$H^*_{C_G(A,c)}(c)_{\mathfrak{p}^N_{(A,c)}} \cong H^*_{C_G(A,c)}(c)_{\mathfrak{p}_{(A,c)}}.$$

We do this by showing that both of the above rings are isomorphic to $H^*_{C_G(A,c)}(c)_{\mathfrak{p}^C_{(A,c)}}$.

This follows from standard results in commutative algebra. Let $R_G = H_G(X)$, $R_N = H_{N_G(A,c)}(c)$, $R_C = H_{C_G(A,c)}(c)$ and $M = H^*_{C_G(A,c)}(c)$. We may consider M as an R_G-, R_N- or R_C-module, using the commutative double triangle:

$$\begin{array}{ccccc} R_G & \longrightarrow & R_N & \longrightarrow & R_C \\ & \searrow & \downarrow & \swarrow & \\ & & M & & \end{array}$$

Now

$$\begin{aligned} M_{\mathfrak{p}} &= M \otimes_{R_G} (R_G)_{\mathfrak{p}} = M \otimes_{R_G} (R_C)\mathfrak{p} \quad \text{(by the main theorem)} \\ &= M \otimes_{R_C} (R_C)_{\mathfrak{p}} \end{aligned}$$

and

$$\begin{aligned} M_{\mathfrak{p}^N} &= M \otimes_{R_N} (R_N)_{\mathfrak{p}^N} = M \otimes_{R_N} (R_C)_{\mathfrak{p}^N} \quad \text{(by the main theorem)} \\ &= M \otimes_{R_C} (R_C)_{\mathfrak{p}^N}. \end{aligned}$$

(Here, $\mathfrak{p} = \mathfrak{p}_{(A,c)}$, $\mathfrak{p}^C = \mathfrak{p}^C_{(A,c)}$ and $\mathfrak{p}^N = \mathfrak{p}^N_{(A,c)}$.) Thus, to prove the theorem it is enough to show that $(R_C)_{\mathfrak{p}} = (R_C)_{\mathfrak{p}^C}$ and $(R_C)_{\mathfrak{p}^N} = (R_C)_{\mathfrak{p}^C}$.

There is a natural map $(R_C)_{\mathfrak{p}} \to (R_C)_{\mathfrak{p}^C}$ since $r^{-1}_{G,C}(\mathfrak{p}^C) = \mathfrak{p}$. This map is an isomorphism if we show that $(R_C)_{\mathfrak{p}}$ is a local ring **[M]**. If $\bar{\mathfrak{q}}$ is *any* prime in $(R_C)_{\mathfrak{p}}$

then $\bar{\mathfrak{q}}$ corresponds to a unique prime ideal $\mathfrak{q}$ of R_C disjoint from $r_{G,C}(R_G - \rho)$ (i.e., $\mathfrak{q}_{\mathfrak{p}} = \bar{\mathfrak{q}}$). Since $\mathfrak{q} \cap r_{G,C}(R_G - \mathfrak{p}) = \varnothing$, $r_{G,C}^{-1}(\mathfrak{q}) \cap R_G - \mathfrak{p} = \varnothing$, so $\mathfrak{p} \supseteq r_{G,C}^{-1}(\mathfrak{q})$. But $\mathfrak{p}$ is *minimal* in R_G (using Quillen's characterization of minimal primes), so $\mathfrak{p} = r_{G,C}^{-1}(\mathfrak{q})$.

Now, $R_G/\mathfrak{p} \to R_C/\mathfrak{p}^C$ is *integral* **[Q1]** and *injective* (since the triangle

$$\begin{array}{ccc} R_G & \longrightarrow & R_C \\ & \searrow \quad \swarrow & \\ & R_A/\sqrt{0} = H_A(\mathrm{pt} \in c)/\sqrt{0} & \end{array}$$

commutes), and both $\mathfrak{q}/\mathfrak{p}^C$ and the 0-ideal in $R_C/\mathfrak{p}^C$ contract to the 0-ideal in $R_G/\mathfrak{p}$; by the going-up theorem (**[M]**, e.g.) we have $\mathfrak{q}/\mathfrak{p}^C = 0$ or $\mathfrak{p}^C = \mathfrak{q}$. Thus, in fact, we have shown that $\bar{\mathfrak{q}} = (\mathfrak{p}^C)_{\mathfrak{p}}$ is the *only* prime in $(R_C)_{\mathfrak{p}}$.

The same argument works for R_N and $\mathfrak{p}^N$ in place of R_G and $\mathfrak{p}$. Q.E.D.

In any noetherian ring the 0-ideal may be decomposed as an intersection of primary ideals: $0 = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$. The set $\{\sqrt{\mathfrak{q}_i}\}$ is the set of associated primes of the ring. The minimal primes of the ring are the minimal elements of this set $\{\sqrt{\mathfrak{q}_i}\}$. The ideals of $\{\mathfrak{q}_i\}$ belonging to these minimal primes are called the *isolated* primary components; the rest are called embedded components. The isolated primary ideals are uniquely specified, the embedded components are not. The following theorem characterizes the isolated primary components of $H_G(X)$.

THEOREM 3.8. *If G is finite, the isolated primary ideals of $H_G(X)$ are of the form*

$$\mathfrak{q}_{(A,c)} = \ker\bigl(H_G(X) \to H_{C_G(A,c)}(c)\bigr),$$

where (A, c) is a maximal pair of $\mathscr{A}(G, X)$.

PROOF. We need only show that if (A, c) is maximal then $\mathfrak{q}_{(A,c)}$ is $\mathfrak{p}_{(A,c)}$-primary; i.e., $\sqrt{\mathfrak{q}_{(A,c)}} = \mathfrak{p}_{(A,c)}$ and $\mathfrak{q}_{(A,c)}$ is primary (in view of Quillen's characterization of the minimal primes).

Consider the map

$$H_G(X) \overset{r_{G,C}}{\to} H_{C_G(A,c)}(c).$$

Let $\mathfrak{p}^C = \mathfrak{p}^C_{(A,c)}$ and $\mathfrak{p} = \mathfrak{p}_{(A,c)}$ be as above. Since $\mathfrak{p}^C$ is the *only* associated prime of $H_{C_G(A,c)}(c)$ (this ring is Cohen-Macaulay by **[D1]** so has no embedded primes, and $\mathfrak{p}^C$ is the only *minimal* prime by **[Q1]**) we see that 0 is $\mathfrak{p}^C$-primary (uniqueness of primary decomposition). Thus $r_{(G,C)}^{-1}(0)$ is $r_{G,C}^{-1}(\mathfrak{p}^C)$-primary—i.e., $\mathfrak{q}_{(A,c)}$ is $\mathfrak{p}$-primary. Q.E.D.

REMARK. This theorem should be true for G compact Lie, but the results of **[D1]** needed are proved only for finite groups at this point.

It is perhaps of interest to see what these theorems say when X is a one-point space. Then $H_G^*(X)$ is group cohomology—$H^*(BG)$. For example, Theorem 3.2 gives an isomorphism

$$H^*(BG)_{\mathfrak{p}_A} \to H^*(B(C_G(A)))_{\mathfrak{p}_A}^{W_G(A)},$$

where $\mathfrak{p}_A = \ker(H^*(BG) \to H^*(BA)/\sqrt{0})$ is the prime belonging to a maximal p-torus A of G. In the next section we give an application of this result.

4. An example. Let $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z}) = \mathrm{GL}_n$ denote the group of invertible $n \times n$ matrices over $\mathbf{Z}/p\mathbf{Z}$; or equivalently the group of invertible linear maps of an n-dimensional vector space over $\mathbf{Z}/p\mathbf{Z}$. Consider the subgroup of GL_n given by

$$A_1 = \left\{ \begin{bmatrix} 1 & a_1 & \cdots & a_{n-1} \\ & 1 & & \bigcirc \\ & & \ddots & \\ & \bigcirc & & 1 \end{bmatrix} \middle| \, a_i \in \mathbf{Z}/p\mathbf{Z},\ i = 1,2,\ldots,n-1 \right\}.$$

This subgroup is a p-torus in GL_n. One can compute its centralizer

$$C_{\mathrm{GL}_n}(A_1) = \left\{ \begin{bmatrix} d & a_1 & \cdots & a_{n-1} \\ & d & & \bigcirc \\ & & \ddots & \\ & \bigcirc & & d \end{bmatrix} \middle| \, d \in (\mathbf{Z}/p\mathbf{Z})^* \right\}$$

$$\cong (\mathbf{Z}/p\mathbf{Z})^* \times A_1$$

(here, $(\mathbf{Z}/p\mathbf{Z})^*$ is the group of unit in $\mathbf{Z}/p\mathbf{Z}$ of order $p-1$), and its normalizer, the parabolic subgroup

$$N_{\mathrm{GL}_n}(A_1) = P_1 = \left\{ \left[\begin{array}{c|ccc} x & a_1 & \cdots & a_{n-1} \\ \hline \bigcirc & & Y & \end{array} \right] \middle| \, x \in (\mathbf{Z}/p\mathbf{Z})^* = \mathrm{GL}_1,\ Y \in \mathrm{GL}_{n-1} \right.$$
$$\left. \text{and } a_i \in \mathbf{Z}/p\mathbf{Z},\ i = 1,2,\ldots,n-1 \right\}.$$

Since the index of A_1 in its centralizer has order prime to p, we see that A_1 is *maximal* p-torus of GL_n. The "Weyl" group of A_1, $W_{\mathrm{GL}_n}(A_1) = P_1/(A_1 \times (\mathbf{Z}/p\mathbf{Z})^*)$ is contained in the automorphism group of A_1, which is naturally isomorphic to GL_{n-1}. One sees that this inclusion $W_{\mathrm{GL}_n}(A_1) \hookrightarrow \mathrm{Aut}(A_1)$ is, in fact, an isomorphism (by computing orders, for example); also, the action of $W_{\mathrm{GL}_n}(A_1)$ on A_1 is equivalent to the natural action of GL_{n-1} on A_1 (where A_1 is regarded as being an $(n-1)$-dimensional vector space over $\mathbf{Z}/p\mathbf{Z}$).

Using the Künneth formula, one computes

$$H^*(C_{\mathrm{GL}_n}(A_1)) = H^*(A_1) \otimes_{\mathbf{Z}/p\mathbf{Z}} H^*((\mathbf{Z}/p\mathbf{Z})^*),$$

since $|(\mathbf{Z}/p\mathbf{Z})^*| = p-1$ is prime to p, its mod-p cohomology is zero in positive degrees, so that

$$H^*(C_{\mathrm{GL}_n}(A_1)) = H^*(A_1).$$

This last ring is well known:

$$H^*(A_1, \mathbf{Z}/p\mathbf{Z}) \cong \begin{cases} \mathbf{Z}/p\mathbf{Z}[a_1,\ldots,a_{n-1}] \otimes_{\mathbf{Z}/p\mathbf{Z}} \bigwedge[b_1,\ldots,b_{n-1}], & p > 2, \\ \mathbf{Z}/p\mathbf{Z}[a_1,\ldots,a_{n-1}], & p = 2 \end{cases}$$

(i.e., either a polynomial algebra tensored with an exterior algebra, or a polynomial algebra).

We assume that p is odd from now on.

Mui [**Mui**] has computed the ring of invariants

$$R^{\mathrm{GL}_{n-1}} = \left(\mathbf{Z}/p\mathbf{Z}[a_1,\dots,a_{n-1}] \otimes_{\mathbf{Z}/p\mathbf{Z}} \bigwedge[b_1,\dots,b_{n-1}]\right)^{\mathrm{GL}_{n-1}};$$

it is generated by $Q_0,\dots,Q_{n-2}$ ($n-1$ polynomial generators) and $R_{s_1,s_2,\dots,s_k}$, where $1 \leqslant k \leqslant n-1$ and $0 \leqslant s_1 < s_2 < \cdots < s_k \leqslant n-2$ is a selection of k numbers from the set $\{0,1,2,\dots,n-2\}$ (thus there are $2^{n-1}-1$ of these generators).

The relations are $R_s^2 = 0$ for every s between 0 and $n-2$ and $R_{s_1}R_{s_2}\cdots R_{s_k} = (-1)^{k(k-1)/2}R_{s_1,\dots,s_k}Q_0^{k-1}$ for every k between 1 and $n-1$, and every subset of k elements $s_1 < s_2 < \cdots < s_k$ of $\{0,1,\dots,n-2\}$.

Since Q_0 is *not* a zero divisor in this invariant subring (it is not a zero divisor in $R \supseteq R^{\mathrm{GL}_{n-1}}$), one sees that the above two relations imply that $R^2_{s_1,\dots,s_k} = 0$ for *every* set of indices $\{s_1,\dots,s_k\}$.

The degrees of the generators are as follows:

$$\operatorname{degree}(Q_i) = 2(p^{n-1}-p^i), \qquad i = 0,1,\dots,n-2,$$

and

$$\operatorname{degree}\left(R_{s_1,\dots,s_k}\right) = k + 2(p^{n-1}-1) - 2(p^{s_1} + \cdots + p^{s_k}).$$

There is a unique minimal prime $\mathfrak{p}$ in the even part of this ring, namely, the nilpotent elements, or the ideal generated by the R's. Localizing at this prime means inverting all the Q_i's. Letting $P_i = R_i/Q_0$ for $0 \leqslant i \leqslant n-2$ ("degree" $(P_i) = 1 - 2p^i$) one sees that the relations in R imply that $P_i^2 = 0$ and that

$$R_{\mathfrak{p}}^{\mathrm{GL}_{n-1}} = \mathbf{Z}/p\mathbf{Z}(Q_0,\dots,Q_{n-2}) \otimes_{\mathbf{Z}/p\mathbf{Z}} \wedge(P_0,\dots,P_{n-2}),$$

i.e., the quotient field of the polynomial ring $\mathbf{Z}/p\mathbf{Z}[Q_0,\dots,Q_{n-2}]$ tensored with an exterior algebra on the P_i's.

Now, since

$$H^*\left(C_{\mathrm{GL}_n}(A_1)\right)_{\mathfrak{p}_{A_1}}^{W_{GL_n}(A_1)} = k(Q_0,\dots,Q_{n-2}) \otimes \wedge(P_0,\dots,P_{n-2})$$

(use the preceding paragraphs and the argument used in the proof of Corollary 3.7), the main theorem of §3 implies that

$$H^*(\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z}),\mathbf{Z}/p\mathbf{Z})_{\mathfrak{p}_{A_1}} \cong k(Q_0,\dots,Q_{n-2}) \otimes \wedge(P_0,\dots,P_{n-2})$$

(here $k = \mathbf{Z}/p\mathbf{Z}$).

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